Distributed Hierarchical Distribution Control for Very-Large-Scale Clustered Multi-Agent Systems (Supplementary Material)

Augustinos D. Saravanos, Yihui Li and Evangelos A. Theodorou
Georgia Institute of Technology, GA, USA
Email: asaravanos@gatech.edu

***Abstract*—This document serves as supplementary material for the main paper. A video with a more detailed illustration of the simulation results is also provided.**

I. DHDE Details

A. Proof of Proposition 1

First, it is straightforward to show the equivalence between minimizing the costs in (11a) and (13a), since $\mathbb{D}_{\mathrm{KL}}(\mathcal{N}_n^{\ell+1} \| \mathcal{N}_i^{\ell})$ is given by

$$\mathbb{D}_{\mathrm{KL}}(\mathcal{N}_n^{\ell+1} \| \mathcal{N}_i^{\ell}) = \frac{1}{2}\Bigg[\log\frac{|\Sigma_i|}{|\Sigma_n|} - n_x + \mathrm{tr}(\Sigma_i^{-1}\Sigma_n) + (\mu_i - \mu_n)^{\mathrm{T}}\Sigma_i^{-1}(\mu_i - \mu_n)\Bigg] \tag{43}$$

which yields $\hat{J}_i^{\mathrm{e}}(Q_i, q_i)$ after substituting with $Q_i = \Sigma_i^{-1}$, $q_i = \Sigma_i^{-1}\mu_i$ and neglecting the constant terms. Note that the objective function $\hat{J}_i^{\mathrm{e}}(Q_i, q_i)$ is jointly convex for Q_i, q_i.

Next, we show the equivalence between the constraints (11b) and (13b-c). For the convenience of the reader, let us first restate a result known as the S-Lemma by Yakubovich [1]. According to the S-Lemma, given two functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ with $f_1(x) = x^{\mathrm{T}}A_1x + b_1^{\mathrm{T}}x + c_1$ and $f_2(x) = x^{\mathrm{T}}A_2x + b_2^{\mathrm{T}}x + c_2$ and if there exists an $\bar{x}$ such that $f_1(\bar{x}) > 0$, then the following is true

$$f_1(x) \geq 0 \Rightarrow f_2(x) \geq 0, \quad \forall x, \tag{44}$$

if and only if there exists a $\tau \geq 0$ such that $f_2(x) \geq \tau f_1(x),\ \forall x$. In constraint (11b), we enforce that if $x \in \mathbb{R}^{n_p}$ is such that

$$(x - \bar{\mu}_n)^{\mathrm{T}}\bar{\Sigma}_n^{-1}(x - \bar{\mu}_n) \leq \alpha, \tag{45}$$

then it should follow that

$$(x - \bar{\mu}_i)^{\mathrm{T}}\bar{\Sigma}_i^{-1}(x - \bar{\mu}_i) \leq \alpha. \tag{46}$$

Using the S-Lemma, this is equivalent with imposing the constraints $\tau_n \geq 0,\ \forall n \in \mathcal{C}_i^{\ell}$ and

$$\alpha - (x - \bar{\mu}_i)^{\mathrm{T}}\bar{\Sigma}_i^{-1}(x - \bar{\mu}_i) \geq \tau\big(\alpha - (x - \bar{\mu}_n)^{\mathrm{T}}\bar{\Sigma}_n^{-1}(x - \bar{\mu}_n)\big),$$

which can be written in matrix form as

$$\hat{x}^{\mathrm{T}}V_n\hat{x} \geq 0, \tag{47}$$

where $\hat{x} = [x; 1]$ and

$$V_n = \begin{bmatrix} V_{11} & V_{12} \\ V_{12}^{\mathrm{T}} & V_{22} \end{bmatrix},$$

with

$$V_{11} = -\bar{\Sigma}_i^{-1} + \tau_n\bar{\Sigma}_n^{-1},\ V_{12} = \bar{\Sigma}_i^{-1}\bar{\mu}_i - \tau_n\bar{\Sigma}_n^{-1}\bar{\mu}_n,$$
$$V_{22} = \alpha - \tau_n\alpha - \bar{\mu}_i^{\mathrm{T}}\bar{\Sigma}_i^{-1}\bar{\mu}_i + \tau_n\bar{\mu}_n^{\mathrm{T}}\bar{\Sigma}_n^{-1}\bar{\mu}_n.$$

By definition, the constraint (47) is equivalent with $V_n \succeq 0$. Furthermore, by applying the Schur complement w.r.t. V_{22}, it follows that $V_n \succeq 0$ is equivalent with $S_n \succeq 0$, where

$$S_n = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12}^{\mathrm{T}} & S_{22} & S_{23} \\ 0^{\mathrm{T}} & S_{23}^{\mathrm{T}} & S_{33} \end{bmatrix}, \tag{48}$$

with

$$S_{11} = V_{11},\ S_{12} = V_{12},\ S_{22} = \alpha - \tau_n\alpha + \tau_n\bar{\mu}_n^{\mathrm{T}}\bar{\Sigma}_n^{-1}\bar{\mu}_n,$$
$$S_{23} = (\bar{\Sigma}_i^{-1}\bar{\mu}_i)^{\mathrm{T}},\ S_{33} = \bar{\Sigma}_i^{-1}.$$

The expressions in (14c) follow after substituting with Q_i and q_i. Finally, it is evident that the constraints (11c) and (13d) are equivalent since $\Sigma_i \succ 0$ if and only if $\Sigma_i^{-1} \succ 0$. Note that all constraints are convex as well. Therefore, the problem presented in Proposition 1 is a convex optimization one.

B. Proof of Proposition 2

The equivalence between the costs (16a) and (17a), as well as the equivalence between the constraints (16b) and (17b)-(17c) follow directly from Proposition 1. Next, we show that if the constraints (17d) and (17e) are satisfied, then the constraint (16c) is also satisfied. In fact, the constraint

$$\mathcal{E}_\theta[\bar{\mu}_i, \bar{\Sigma}_i] \cap \mathcal{E}_\theta[\bar{\mu}_j, \bar{\Sigma}_j] = \emptyset \tag{49}$$

will hold if the following constraint holds

$$\mathscr{C}\big[\mathcal{E}_\theta[\bar{\mu}_i, \bar{\Sigma}_i]\big] \cap \mathscr{C}\big[\mathcal{E}_\theta[\bar{\mu}_j, \bar{\Sigma}_j]\big] = \emptyset, \tag{50}$$

where $\mathscr{C}[\mathcal{E}]$ denotes the minimum area enclosing circle of an ellipse $\mathcal{E}$. Of course, $\mathscr{C}\big[\mathcal{E}_\theta[\bar{\mu}_i, \bar{\Sigma}_i]\big]$ is a circle with center $\bar{\mu}_i$ and radius $\sqrt{\alpha\lambda_{\max}(\bar{\Sigma}_i)}$, which is the major axis length of $\mathcal{E}_\theta[\bar{\mu}_i, \bar{\Sigma}_i]$. Hence, the constraint (50) can be rewritten as

$$\|\bar{\mu}_i - \bar{\mu}_j\|_2 \geq \sqrt{\alpha\lambda_{\max}(\bar{\Sigma}_i)} + \sqrt{\alpha\lambda_{\max}(\bar{\Sigma}_j)}. \tag{51}$$

or equivalently as

$$\|\bar{Q}_i^{-1}\bar{q}_i - \bar{Q}_j^{-1}\bar{q}_j\|_2 \geq \frac{\sqrt{\alpha}}{\sqrt{\lambda_{\min}(\bar{Q}_i)}} + \frac{\sqrt{\alpha}}{\sqrt{\lambda_{\min}(\bar{Q}_j)}}. \quad (52)$$

By introducing the auxiliary variables ϕ_i, ϕ_j, the constraint (52) is equivalent with the set of constraints

$$\|\bar{Q}_i^{-1}\bar{q}_i - \bar{Q}_j^{-1}\bar{q}_j\|_2 \geq \phi_i^{-1/2} + \phi_j^{-1/2}, \quad (53a)$$

$$\phi_l^{-1/2} \geq \frac{\sqrt{\alpha}}{\sqrt{\lambda_{\min}(\bar{Q}_l)}},\ l \in \{i, j\} \quad (53b)$$

where (53a) is the same as (17d). The constraint (53b) can be rewritten as

$$\bar{Q}_l \succeq \phi_l \alpha I \quad (54)$$

which yields (17e). Finally, the constraints (16d) and (17f) are equivalent.

C. ADMM Derivation

After introducing the augmented variables $\tilde{Q}_i$, $\tilde{q}_i$, $\tilde{\phi}_i$, and the global ones G, g, z, the problem presented in Proposition 2 can be reformulated as

$$\min \sum_{i \in \mathcal{C}_a^{\ell-1}} \hat{J}_i^{\mathrm{e}}(Q_i, q_i) \quad (55a)$$

$$\begin{aligned}
\text{s.t}\quad & S_{i,n}(Q_i, q_i, \tau_{i,n}) \succeq 0,\ n \in \mathcal{C}_i^{\ell}, & (55b)\\
& \tau_{i,n} \geq 0,\ n \in \mathcal{C}_i^{\ell}, & (55c)\\
& h_i(\tilde{Q}_i, \tilde{q}_i, \tilde{\phi}_i) \leq 0, & (55d)\\
& T_i(Q_i, \phi_i) \succeq 0, & (55e)\\
& Q_i \succeq 0, & (55f)\\
& \tilde{Q}_i = \tilde{G}_i,\ \tilde{q}_i = \tilde{g}_i,\ \tilde{\phi}_i = \tilde{z}_i,\ i \in \mathcal{C}_a^{\ell-1}, & (55g)
\end{aligned}$$

where $h_i(\tilde{Q}_i, \tilde{q}_i, \tilde{\phi}_i)$ is defined as

$$h_i = [\{h_{i,j}(Q_i, q_i, \phi_i, Q_j^i, q_j^i, \phi_j^i)\}_{j \in \mathfrak{n}[\mathcal{C}_i^{\ell}]}]. \quad (56)$$

Let us also introduce the indicator functions $\mathcal{I}_{S_i}(Q_i, q_i, \tau_{i,n})$, $\mathcal{I}_{\tau_{i,n}}(\tau_{i,n})$, $\mathcal{I}_{h_i}(\tilde{Q}_i, \tilde{q}_i, \tilde{\phi}_i)$, $\mathcal{I}_{T_i}(Q_i, \phi_i)$, $\mathcal{I}_{Q_i}(Q_i)$, which take a zero value if the constraints (55b), (55c), (55d), (55e), (55f), respectively, are satisfied, and become infinite, otherwise. Then, the Augmented Lagrangian (AL) for this problem can be formulated as

$$\begin{aligned}
\mathcal{L} = \sum_{i \in \mathcal{C}_a^{\ell-1}} & \hat{J}_i^{\mathrm{e}}(Q_i, q_i) + \mathcal{I}_{h_i}(\tilde{Q}_i, \tilde{q}_i, \tilde{\phi}_i) + \mathcal{I}_{T_i}(Q_i, \phi_i)\\
& + \mathcal{I}_{Q_i}(Q_i) + \sum_{n \in \mathcal{C}_i^{\ell}} \mathcal{I}_{S_{i,n}}(Q_i, q_i, \tau_{i,n}) + \mathcal{I}_{\tau_{i,n}}(\tau_{i,n})\\
& + \mathrm{tr}(\Xi_i^{\mathrm{T}}(\tilde{Q}_i - \tilde{G}_i)) + \xi_i^{\mathrm{T}}(\tilde{q}_i - \tilde{g}_i) + y_i^{\mathrm{T}}(\tilde{\phi}_i - \tilde{z}_i)\\
& + \frac{\rho_Q}{2}\|\tilde{Q}_i - \tilde{G}_i\|_F^2 + \frac{\rho_q}{2}\|\tilde{q}_i - \tilde{g}_i\|_2^2 + \frac{\rho_\phi}{2}\|\tilde{\phi}_i - \tilde{z}_i\|_2^2.
\end{aligned}$$

Therefore, the ADMM updates are derived as follows. First, the updates for the variables $\tilde{Q}_i$, $\tilde{q}_i$ and $\tilde{\phi}_i$, are given by

$$\{\tilde{Q}_i, \tilde{q}_i, \tilde{\phi}_i\} = \operatorname{argmin} \mathcal{L} \quad (57)$$

for all $i \in \mathcal{C}_a^{\ell-1}$. The minimization in (57) leads to the local problems

$$\begin{aligned}
& \{\tilde{Q}_i, \tilde{q}_i, \tilde{\phi}_i\} = \operatorname{argmin} \tilde{J}_i^{\mathrm{e}}(\tilde{Q}_i, \tilde{q}_i, \tilde{\phi}_i) & (58a)\\
\text{s.t}\quad & S_{i,n}(Q_i, q_i, \tau_{i,n}) \succeq 0,\ n \in \mathcal{C}_i^{\ell}, & (58b)\\
& \tau_{i,n} \geq 0,\ n \in \mathcal{C}_i^{\ell}, & (58c)\\
& h_i(\tilde{Q}_i, \tilde{q}_i, \tilde{\phi}_i) \leq 0, & (58d)\\
& T_i(Q_i, \phi_i) \succeq 0, & (58e)\\
& Q_i \succ 0, & (58f)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{J}_i^{\mathrm{e}} = \hat{J}_i^{\mathrm{e}}(Q_i, q_i) &+ \mathrm{tr}(\Xi_i^{\mathrm{T}}(\tilde{Q}_i - \tilde{G}_i)) + \xi_i^{\mathrm{T}}(\tilde{q}_i - \tilde{g}_i)\\
&+ y_i^{\mathrm{T}}(\tilde{\phi}_i - \tilde{z}_i) + \frac{\rho_Q}{2}\|\tilde{Q}_i - \tilde{G}_i\|_F^2 + \frac{\rho_q}{2}\|\tilde{q}_i - \tilde{g}_i\|_2^2\\
&+ \frac{\rho_\phi}{2}\|\tilde{\phi}_i - \tilde{z}_i\|_2^2. \quad (59)
\end{aligned}$$

Subsequently, the global variables G, g and z are updated by

$$\{G, g, z\} = \operatorname{argmin} \mathcal{L}. \quad (60)$$

using the updated values of $\tilde{Q}_i$, $\tilde{q}_i$ and $\tilde{\phi}_i$, $\forall i \in \mathcal{C}_a^{\ell-1}$. The minimization in (60) can be separated for all G_i, g_i and z_i, leading to the following averaging steps

$$G_i = \frac{1}{|\mathfrak{m}'[\mathcal{C}_i^{\ell}]|} \sum_{j \in \mathfrak{m}'[\mathcal{C}_i^{\ell}]} Q_i^j \quad (61a)$$

$$g_i = \frac{1}{|\mathfrak{m}'[\mathcal{C}_i^{\ell}]|} \sum_{j \in \mathfrak{m}'[\mathcal{C}_i^{\ell}]} q_i^j \quad (61b)$$

$$z_i = \frac{1}{|\mathfrak{m}'[\mathcal{C}_i^{\ell}]|} \sum_{j \in \mathfrak{m}'[\mathcal{C}_i^{\ell}]} \phi_i^j. \quad (61c)$$

After these updates are performed, then the dual variables are updated through dual ascent steps, as follows

$$\Xi_i \leftarrow \Xi_i + \rho_Q(\tilde{Q}_i - \tilde{G}_i) \quad (62a)$$

$$\xi_i \leftarrow \xi_i + \rho_q(\tilde{q}_i - \tilde{g}_i) \quad (62b)$$

$$y_i \leftarrow y_i + \rho_\phi(\tilde{\phi}_i - \tilde{z}_i), \quad (62c)$$

by all $i \in \mathcal{C}_a^{\ell-1}$.

D. Implementation Details

1) Constraint Linearization: In the local problems (21), all cost terms and constraints are convex, except for the constraint (21d). We accommodate for that by linearizing the constraint in every ADMM iteration around the previous values of the included variables, which we denote with $\bar{Q}_i', \bar{q}_i', \phi_i', \bar{Q}_j{}', \bar{q}_j{}', \phi_j{}'$, where we drop the superscript i to lighten the notation. The first order Taylor approximation of $h_{i,j}$ around $(\bar{Q}_i', \bar{q}_i', \phi_i', \bar{Q}_j{}', \bar{q}_j{}', \phi_j{}')$ is denoted by

$\bar{h}_{i,j}(\bar{Q}_i, \bar{q}_i, \phi_i, \bar{Q}_j, \bar{q}_j, \phi_j)$ where

$$\begin{aligned}\bar{h}_{i,j} &= h_{i,j}(\bar{Q}'_i, \bar{q}'_i, \phi'_i, \bar{Q}'_j, \bar{q}'_j, \phi'_j) + \mathrm{tr}\left(\nabla_{\bar{Q}_i} h_{i,j}\Big|^{\mathrm{T}}_{\bar{Q}'_i}(\bar{Q}_i - \bar{Q}'_i)\right) \\ &+ \mathrm{tr}\left(\nabla_{\bar{Q}_j} h_{i,j}\Big|^{\mathrm{T}}_{\bar{Q}'_j}(\bar{Q}_j - \bar{Q}'_j)\right) + \nabla_{\bar{q}_i} h_{i,j}\Big|^{\mathrm{T}}_{\bar{q}'_i}(\bar{q}_i - \bar{q}'_i) \\ &+ \nabla_{\bar{q}_j} h_{i,j}\Big|^{\mathrm{T}}_{\bar{q}'_j}(\bar{q}_j - \bar{q}'_j) + \frac{\partial h_{i,j}}{\partial \phi_j}\Big|_{\phi'_j}(\phi_j - \phi'_j) \\ &+ \frac{\partial h_{i,j}}{\partial \phi_i}\Big|_{\phi'_i}(\phi_i - \phi'_i),\end{aligned}$$

with

$$\begin{aligned}\nabla_{\bar{Q}_i} h_{i,j} &= \frac{1}{\|\omega_{i,j}\|_2}\bar{Q}_i^{-\mathrm{T}}\omega_{i,j}(\bar{Q}_i^{-1}\bar{q}_i)^{\mathrm{T}} \\ \nabla_{\bar{Q}_j} h_{i,j} &= -\frac{1}{\|\omega_{i,j}\|_2}\bar{Q}_j^{-\mathrm{T}}\omega_{i,j}(\bar{Q}_j^{-1}\bar{q}_j)^{\mathrm{T}} \\ \nabla_{\bar{q}_i} h_{i,j} &= -\frac{1}{\|\omega_{i,j}\|_2}\bar{Q}_i^{-\mathrm{T}}\omega_{i,j} \\ \nabla_{\bar{q}_j} h_{i,j} &= \frac{1}{\|\omega_{i,j}\|_2}\bar{Q}_j^{-\mathrm{T}}\omega_{i,j} \\ \omega_{i,j} &= \bar{Q}_i^{-1}\bar{q}_i - \bar{Q}_j^{-1}\bar{q}_j, \\ \frac{\partial h_{i,j}}{\partial \phi_i} &= -\frac{1}{2}\phi_i^{-3/2}, \quad \frac{\partial h_{i,j}}{\partial \phi_j} = -\frac{1}{2}\phi_j^{-3/2}.\end{aligned}$$

2) Termination Criterion: We suggest two options for the termination criterion in Line 10 of Alg. 1. The first one that would not require any additional communication would be to just set a maximum amount of ADMM iterations. The second option would be to also check whether the ADMM primal and dual residuals norms are below some prespecified thresholds to allow for early termination. In particular, the primal residuals norms are given by

$$\begin{aligned}\epsilon_{\mathrm{primal},1} &= \sum_{i \in \mathcal{C}_a^{\ell-1}} \|\tilde{Q}_i - \tilde{T}_i\|_F, \\ \epsilon_{\mathrm{primal},2} &= \sum_{i \in \mathcal{C}_a^{\ell-1}} \|\tilde{q}_i - \tilde{t}_i\|_2, \\ \epsilon_{\mathrm{primal},3} &= \sum_{i \in \mathcal{C}_a^{\ell-1}} \|\tilde{\phi}_i - \tilde{z}_i\|_2,\end{aligned}$$

while the dual residuals norms are given by

$$\begin{aligned}\epsilon_{\mathrm{dual},1} &= \rho_Q \sum_{i \in \mathcal{C}_a^{\ell-1}} \|\tilde{T}_i - \tilde{T}_{i,\mathrm{prev}}\|_F, \\ \epsilon_{\mathrm{dual},2} &= \rho_q \sum_{i \in \mathcal{C}_a^{\ell-1}} \|\tilde{t}_i - \tilde{t}_{i,\mathrm{prev}}\|_2, \\ \epsilon_{\mathrm{dual},3} &= \rho_\phi \sum_{i \in \mathcal{C}_a^{\ell-1}} \|\tilde{z}_i - \tilde{z}_{i,\mathrm{prev}}\|_2.\end{aligned}$$

Note that the latter approach would require all agents $i \in \mathcal{C}_a^{\ell-1}$ sending their variables to agent a so that the residuals are computed.

II. DHDS DETAILS

A. Detailed Expressions

The decision variables $\bar{u}_i \in \mathbb{R}^{Nn_u}$, $L_i \in \mathbb{R}^{Nn_u \times n_x}$ and $K_i \in \mathbb{R}^{Nn_u \times Nn_x}$ are given by $\bar{u}_i = [\bar{u}_{i,0}; \dots; \bar{u}_{i,N-1}]$,

$$\bar{u}_i = \begin{bmatrix}\bar{u}_{i,0}^{\mathrm{T}} & \bar{u}_{i,1}^{\mathrm{T}} & \cdots & \bar{u}_{i,N-1}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}},$$

$$L_i = \begin{bmatrix}L_{i,0}^{\mathrm{T}} & L_{i,1}^{\mathrm{T}} & \cdots & L_{i,N-1}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}},$$

$$K_i = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ K_{i,(0,0)} & 0 & \dots & 0 & 0 \\ K_{i,(1,0)} & K_{i,(1,1)} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{i,(N-2,0)} & K_{i,(N-2,1)} & \dots & K_{i,(N-2,N-2)} & 0 \end{bmatrix}.$$

The matrices Ψ_0, Ψ_u and Ψ_w have the following form

$$\Psi_0 = \begin{bmatrix} I & A^{\mathrm{T}} & \cdots & A^{N\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

$$\Psi_u = \begin{bmatrix} 0 & 0 & \dots & 0 \\ B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix},$$

$$\Psi_w = \begin{bmatrix} 0 & 0 & \dots & 0 \\ I & 0 & \cdots & 0 \\ A & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1} & A^{N-2} & \cdots & I \end{bmatrix}.$$

The mean state $\mu_{i,k}$ is given by

$$\mu_{i,k} = f_{i,k}(\bar{u}_i) = P_k f_i(\bar{u}_i), \tag{63}$$

where

$$f_i(\bar{u}_i) = \Psi_0 \mu_{i,0} + \Psi_u \bar{u}_i, \tag{64}$$

and $P_k := \begin{bmatrix}0, \dots, I, \dots, 0\end{bmatrix} \in \mathbb{R}^{N_x \times (N+1)N_x}$. Furthermore, the state covariance $\Sigma_{i,k}$ is given by

$$\Sigma_{i,k} = F_{i,k}(L_i, K_i) = P_k F_i(L_i, K_i) P_k^{\mathrm{T}}, \tag{65}$$

where

$$\begin{aligned}F_i(L_i, K_i) := {}&(\Psi_0 + \Psi_u L_i)\Sigma_{i,0}(\Psi_0 + \Psi_u L_i)^{\mathrm{T}} \\ &+ (\Psi_w + \Psi_u K_i)\bar{W}(\Psi_w + \Psi_u K_i)^{\mathrm{T}}.\end{aligned}$$

with $\bar{W} = \mathrm{blkdiag}(W, \dots, W) \in \mathbb{S}^{+}_{Nn_x}$.

B. Proof of Proposition 3

First, let us show the equivalence between costs (25a) and (32a). The cost function $J_i^{\mathrm{s}}(u_i^{\ell})$ can be rewritten as

$$J_i^{\mathrm{s}}(u_i) = \mathbb{E}[u_i^{\mathrm{T}}\bar{R}u_i] = \mathbb{E}\big[\mathrm{tr}(\bar{R}_i u_i u_i^{\mathrm{T}})\big] = \mathrm{tr}\big(\mathbb{E}[\bar{R}_i u_i u_i^{\mathrm{T}}]\big).$$

Using (28), we obtain

$$\begin{aligned}J_i^{\mathrm{s}}(u_i) &= \hat{J}_i^{\mathrm{s}}(\bar{u}_i, L_i, K_i)\\ &= \mathrm{tr}\big(\mathbb{E}[\bar{R}_i(\bar{u}_i + L_i\tilde{x}_{i,0} + K_i w_i)(\bar{u}_i + L_i\tilde{x}_{i,0} + K_i w_i)^{\mathrm{T}}]\big)\\ &= \mathrm{tr}(\bar{R}\bar{u}_i\bar{u}_i^{\mathrm{T}} + \bar{R}K_iWK_i^{\mathrm{T}} + \bar{R}L_i\Sigma_{i,0}L_i^{\mathrm{T}})\\ &= \bar{u}_i^{\mathrm{T}}\bar{R}\bar{u}_i + \mathrm{tr}(\bar{R}K_iWK_i^{\mathrm{T}} + \bar{R}L_i\Sigma_{i,0}L_i^{\mathrm{T}}),\end{aligned}$$

where $\tilde{x}_{i,0} = x_{i,0} - \mu_{i,0}$ and we used the facts that $\mathbb{E}[\tilde{x}_{i,0}] = 0$, $\mathbb{E}[\tilde{x}_{i,0}w_i^{\mathrm{T}}] = 0$, $\mathbb{E}[w_iw_i^{\mathrm{T}}] = \bar{W}$ and $\mathbb{E}[\tilde{x}_{i,0}\tilde{x}_{i,0}^{\mathrm{T}}] = \Sigma_{i,0}$. Furthermore, the dynamics constraints (25b) are implicitly satisfied since in all expressions we use (31) for the state mean and covariance.

It is also trivial to show that the constraint $\mathfrak{f}_{i,N}(\bar{u}_i) = 0$ is equivalent to $\mathbb{E}[x_{i,N}] = \mu_{i,\mathrm{f}}$. Moreover, if we write $F_i(L_i, K_i) = \Phi_i(L_i, K_i)\Phi_i(L_i, K_i)^{\mathrm{T}}$ with

$$\Phi_i(L_i, K_i) = \left[(\Psi_0 + \Psi_u L_i) \quad (\Psi_w + \Psi_u K_i)\right]\Omega_i, \tag{66}$$

where $\Omega_i\Omega_i^{\mathrm{T}} = \mathrm{blkdiag}(\Sigma_{i,0}, W)$ and define $\Phi_{i,k}(L_i, K_i) = P_k\Phi_i(L_i, K_i)$, then the constraint $\Sigma_{i,\mathrm{f}} \succeq F_{i,N}(L_i, K_i) = \Phi_{i,k}(L_i, K_i)\Phi_{i,k}(L_i, K_i)^{\mathrm{T}}$ is equivalent with

$$\mathcal{F}_{i,N}(L_i, K_i) = \begin{bmatrix}\Sigma_{i,\mathrm{f}} & \Phi_{i,N}(L_i, K_i)\\ \Phi_{i,N}(L_i, K_i)^{\mathrm{T}} & I\end{bmatrix} \succeq 0 \tag{67}$$

by using the Schur complement of $\mathcal{F}_{i,N}(L_i, K_i)$ w.r.t. I.

Subsequently, we show that if the constraints $\mathcal{Q}_{i,k}(L_i, K_i) \succeq 0$ and $\mathfrak{q}_{i,j,k}(\bar{u}_i, \bar{u}_j) \geq 0$ are satisfied, then the constraint $q_{i,j,k}(p_{i,k}, p_{j,k}) \geq 0$ is satisfied as well. In particular, the latter constraint will be true if the following inequalities hold,

$$\|\bar{\mu}_{i,k} - \bar{\mu}_{j,k}\|_2 \geq d_{\mathrm{inter}} + 2r, \tag{68a}$$

$$\sqrt{\alpha\lambda_{\max}(\bar{\Sigma}_{i,k})} \leq r, \tag{68b}$$

where we drop the superscripts ℓ for notational convenience. If we plug the mean state expressions into (68a), then we obtain

$$\|\bar{f}_{i,k}(\bar{u}_i) - \bar{f}_{j,k}(\bar{u}_j)\|_2 \geq d_{\mathrm{inter}} + 2r, \tag{69}$$

which yields the constraint $\mathfrak{q}_{i,j,k}(\bar{u}_i, \bar{u}_j) \geq 0$. Furthermore, the constraint (68b) can be rewritten as

$$\lambda_{\max}(\bar{\Sigma}_{i,k}) \leq \frac{r^2}{\alpha}. \tag{70}$$

which is equivalent with

$$HF_{i,k}(L_i, K_i)H^{\mathrm{T}} - \frac{r^2}{\alpha} \preceq 0. \tag{71}$$

or using again the Schur complement with

$$\begin{bmatrix}\left(\frac{r^\ell}{\sqrt{\alpha}}\right)^2 I & H\Phi_{i,k}(L_i, K_i)\\ \Phi_{i,k}(L_i, K_i)^{\mathrm{T}}H^{\mathrm{T}} & I\end{bmatrix} \succeq 0 \tag{72}$$

which is identical with $\mathcal{Q}_{i,k}(L_i, K_i) \succeq 0$. With similar arguments, it can be shown that if the constraints $\mathcal{Q}_{i,k}(L_i, K_i) \succeq 0$ and $\mathfrak{s}_{i,k}(\bar{u}_i) \geq 0$ are satisfied, then the constraint $s_{i,k}(p_{i.k}) \geq 0$ is also satisfied.

Finally, we wish to show that if the constraint $\mathfrak{p}(\bar{u}_i) \leq 0$ is true, then the constraint (25f) is also true. Since (68b) holds, then it suffices to enforce a contraint that $\bar{\mu}_{i,k}^{\ell}$ should lie within an ellipse with center $\bar{\mu}_{a,k}^{(\ell-1)}$, major axis length $\sqrt{\alpha\lambda_{\max}(\bar{\Sigma}_{a,k}^{(\ell-1)})} - r$, minor axis length $\sqrt{\alpha\lambda_{\min}(\bar{\Sigma}_{a,k}^{(\ell-1)})} - r$, and the same orientation as the ellipse $\mathcal{E}_\theta[\bar{\mu}_{a,k}^{\ell-1}, \bar{\Sigma}_{a,k}^{\ell-1}]$. These specifications can be captured if the following inequality holds

$$(\bar{\mu}_{i,k}^{\ell} - \bar{\mu}_{a,k}^{\ell-1})^{\mathrm{T}}\hat{P}(\bar{\mu}_{i,k}^{\ell} - \bar{\mu}_{a,k}^{\ell-1}) \leq 1, \tag{73}$$

where

$$\hat{P} = \frac{1}{\alpha}U\hat{\Lambda}^{-1}U^{\mathrm{T}},$$

$$\hat{\Lambda} = \left(\Lambda^{1/2} - \frac{r}{\sqrt{\alpha}}I\right)^2,$$

and $[\Lambda, U]$ is the eigendecomposition of $\bar{\Sigma}_{a,k}^{\ell-1}$. This is true since the ellipse $\hat{P}$ and $\bar{\Sigma}_{a,k}^{\ell-1}$ have the same eigenvectors, the major axis length of the ellipse in (73) is

$$\begin{aligned}\sqrt{\frac{1}{\lambda_{\min}(\hat{P})}} &= \sqrt{\alpha\frac{1}{\lambda_{\min}(\hat{\Lambda}^{-1})}} = \sqrt{\alpha\lambda_{\max}(\hat{\Lambda})}\\ &= \sqrt{\alpha}\lambda_{\max}\Big(\Lambda^{1/2} - \frac{r}{\sqrt{\alpha}}I\Big)\\ &= \sqrt{\alpha}\Big(\lambda_{\max}\big(\Lambda^{1/2}\big) - \frac{r}{\sqrt{\alpha}}\Big)\\ &= \sqrt{\alpha\lambda_{\max}(\bar{\Sigma}_{a,k}^{(\ell-1)})} - r\end{aligned}$$

and similarly it can be shown that the minor axis length is

$$\sqrt{\frac{1}{\lambda_{\max}(\hat{P})}} = \sqrt{\alpha\lambda_{\min}(\bar{\Sigma}_{a,k}^{(\ell-1)})} - r.$$

C. ADMM Derivation

The derivation is similar with the one in Section I-C of the SM. With the introduction of the augmented variables $\tilde{u}_i$ and global variable b, problem (35) can be reformulated as

$$\{\bar{u}_i\}_{i\in\mathcal{C}_a^{\ell-1}} = \operatorname{argmin} \sum_{i\in\mathcal{C}_a^{\ell-1}} \hat{J}_{i,1}^{\mathrm{s}}(\bar{u}_i) \tag{74a}$$

$$\text{s.t.}\quad \mathfrak{f}_{i,N}(\bar{u}_i) = 0,\ \mathfrak{s}_{i,k}(\bar{u}_i) \geq 0,\ \mathfrak{p}_{i,k}(\bar{u}_i) \leq 0, \tag{74b}$$

$$\mathfrak{q}_{i,k}(\tilde{u}_i) \geq 0, \quad k \in [\![0, N]\!], \tag{74c}$$

$$\tilde{u}_i = \tilde{b}_i,\ i \in \mathcal{C}_a^{\ell-1}, \tag{74d}$$

with $h_i = [\{h_{i,j}(\bar{u}_i, \bar{u}_j^i)\}_{j\in\mathfrak{n}[\mathcal{C}_i^\ell]}]$. The AL for this problem is given by

$$\begin{aligned}\mathcal{L} = \sum_{i\in\mathcal{C}_a^{\ell-1}} &\hat{J}_i^{\mathrm{s}}(\bar{u}_i) + \mathcal{I}_{\mathfrak{f}_i}(\bar{u}_i) + \mathcal{I}_{\mathfrak{s}_i}(\bar{u}_i) + \mathcal{I}_{\mathfrak{p}_i}(\bar{u}_i)\\ &+ \mathcal{I}_{\mathfrak{q}_i}(\tilde{u}_i) + v_i^{\mathrm{T}}(\tilde{u}_i - \tilde{b}_i) + \frac{\rho_u}{2}\|\tilde{u}_i - \tilde{b}_i\|_2^2,\end{aligned}$$

where the indicator functions are of the same form as in Section I-C of the SM. The updates for the variables $\tilde{u}_i$, are given by $\tilde{u}_i = \operatorname{argmin}\mathcal{L}$, which leads to the local problems

$$\tilde{u}_i = \operatorname{argmin} \tilde{J}_{i,1}^{\mathrm{s}}(\tilde{u}_i) \tag{75a}$$

$$\text{s.t.}\quad \mathfrak{f}_{i,N}(\bar{u}_i) = 0,\ \mathfrak{s}_{i,k}(\bar{u}_i) \geq 0,\ \mathfrak{p}_{i,k}(\bar{u}_i) \leq 0, \tag{75b}$$

$$\mathfrak{q}_{i,j,k}(\bar{u}_i, \bar{u}_j^i) \geq 0,\ j \in \mathfrak{n}[\mathcal{C}_i^\ell],\ k \in [\![0, N]\!], \tag{75c}$$

with

$$\tilde{J}^{\mathrm{s}}_{i,1}(\tilde{u}_i) = \hat{J}^{\mathrm{s}}_{i,1}(\bar{u}_i) + v_i^{\mathrm{T}}(\tilde{u}_i - \tilde{b}_i) + \frac{\rho_u}{2}\|\tilde{u}_i - \tilde{b}_i\|_2^2. \tag{76}$$

The global update given by $b = \operatorname{argmin} \mathcal{L}$, leads to the update rules

$$b_i = \frac{1}{|\mathfrak{m}'[\mathcal{C}_i^\ell]|} \sum_{j \in \mathfrak{m}'[\mathcal{C}_i^\ell]} \bar{u}_i^j \tag{77}$$

using the updated values of $\bar{u}_i^j$. Finally, the dual updates are given by

$$v_i \leftarrow v_i + \rho_u(\tilde{u}_i - \tilde{b}_i). \tag{78}$$

D. Implementation Details

1) Constraint Linearization: In problems (36), all cost terms and constraints are convex, except for the constraints $\mathfrak{q}_{i,j,k}(\bar{u}_i, \bar{u}_j) \geq 0$ and $\mathfrak{s}_{i,k}(\bar{u}_i) \geq 0$. To address these non-convexities, we linearize the constraints in every ADMM iteration around $\bar{u}_i', \bar{u}_j'$, which are the previous values of $\bar{u}_i, \bar{u}_j$. Thus, we replace the aforementioned constraints with

$$\bar{\mathfrak{q}}_{i,j,k}(\bar{u}_i, \bar{u}_j^i) \geq 0, \;\; j \in \mathfrak{n}[\mathcal{C}_i^\ell], \;\; k \in [\![0, N]\!], \tag{79a}$$

$$\bar{\mathfrak{s}}_{i,k}(\bar{u}_i) \geq 0, \;\; k \in [\![0, N]\!], \tag{79b}$$

where

$$\begin{aligned}
\bar{\mathfrak{q}}_{i,j,k}(\bar{u}_i, \bar{u}_j) &= \mathfrak{q}_{i,j,k}(\bar{u}_i', \bar{u}_j') + \nabla_{\bar{u}_i}\mathfrak{q}_{i,j,k}\Big|_{\bar{u}_i'}^{\mathrm{T}}(\bar{u}_i - \bar{u}_i') \\
&\quad + \nabla_{\bar{u}_j}\mathfrak{q}_{i,j,k}\Big|_{\bar{u}_j'}^{\mathrm{T}}(\bar{u}_j - \bar{u}_j'), \\
\bar{\mathfrak{s}}_{i,k}(\bar{u}_i) &= \mathfrak{s}_{i,k}(\bar{u}_i') + \nabla_{\bar{u}_i}\mathfrak{s}_{i,k}\Big|_{\bar{u}_i'}^{\mathrm{T}}(\bar{u}_i - \bar{u}_i')
\end{aligned}$$

and

$$\begin{aligned}
\nabla_{\bar{u}_i}\mathfrak{q}_{i,j,k} &= \frac{1}{\|\zeta_{i,j,k}\|_2}(HP_k\Psi_u)^{\mathrm{T}}\zeta_{i,j,k} \\
\nabla_{\bar{u}_j}\mathfrak{q}_{i,j,k} &= -\frac{1}{\|\zeta_{i,j,k}\|_2}(HP_k\Psi_u)^{\mathrm{T}}\zeta_{i,j,k} \\
\zeta_{i,j,k} &= HP_k\big(\Psi_0(\mu_{i,0} - \mu_{j,0}) + \Psi_u(\bar{u}_i - \bar{u}_j)\big) \\
\nabla_{\bar{u}_i}\mathfrak{s}_{i,k} &= \frac{1}{\|\eta_{i,k}\|_2}(HP_k\Psi_u)^{\mathrm{T}}\eta_{i,k} \\
\eta_{i,k} &= HP_k\big(\Psi_0\mu_{i,0} + \Psi_u\bar{u}_i\big) - p_o.
\end{aligned}$$

2) Termination Criterion: The termination criterion in Line 10 of Alg. 2 is similar with one presented in Section I-D of the SM. In particular, we either set a maximum amount of ADMM iterations or check whether the residual norms

$$\begin{aligned}
\epsilon_{\mathrm{primal}} &= \sum_{i \in \mathcal{C}_a^{\ell-1}} \|\tilde{u}_i - \tilde{b}_i\|_2, \\
\epsilon_{\mathrm{dual}} &= \rho_u \sum_{i \in \mathcal{C}_a^{\ell-1}} \|\tilde{b}_i - \tilde{b}_{i,\mathrm{prev}}\|_2,
\end{aligned}$$

are below some predefined thresholds. Note that in the latter case, all agents $i \in \mathcal{C}_a^{\ell-1}$ would be required to send their variables to agent a during every ADMM iteration.

III. Simulation Details

In the simulation experiments, all agents are modeled with 2D double integrator dynamics which are discretized with $dt = 0.05s$. The time horizon is $N = 100$ for all tasks. For the first two tasks, the noise covariance is $W = \operatorname{diag}(0.02, 0.02, 0.2, 0.2)^2$, while for the third one it is $W = \operatorname{diag}(10^{-3}, 10^{-3}, 10^{-2}, 10^{-2})^2$. For all tasks, we set $\theta = 0.997$. For both DHDE and DHDS, the maximum amount of ADMM iterations is set to 20. All penalty parameters are selected to be $\rho = 10^3$.

References

[1] Vladimir Andreevich Yakubovich. The S-procedure in non-linear control theory. *Vestnik Leningrad Univ. Mathe.*, 4:73–93, 1977.